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Odsjek: Matematika i informatika
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Linearna algebra, majski apsolventski pismeni

1. Dokazati da je $\mathcal{V} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \,|\, \mathrm{trag}(A) = 0\}$ vektorski podprostor prostora $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ (gdje je $\mathrm{trag}(A) =$ suma dijagonalnih elemenata matrice A). Odrediti mu bazu i dimenziju. Nadopunite nađenu bazu do baze za $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

2. Odrediti za koje vrijednosti nepoznate x će vektor $(0,1,1,4)^\top \in \mathbb{R}^4$ pripadati $\mathrm{im}(A)$ ako je

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1-x & 1 & 1 & 1 \\ 0 & 1-x & 1 & 1 \\ 0 & 0 & 1-x & 1 \end{bmatrix}.$$

3. Zadano je preslikavanje $T : V^3(0) \longrightarrow V^3(0)$ izrazom

$$T(a\vec{i} + b\vec{j} + c\vec{k}) = (a - 2b + c)\vec{i} + 3a\vec{j} - (2a - 4c)\vec{k}.$$

Dokazati da je T linearni operator i odredite mu matrični prikaz u bazi $\mathcal{B} = \{\vec{i} - \vec{j}, 2\vec{i} + \vec{j}, \vec{i} + \vec{k}\}$ (drugim riječima odredite $[T]_\mathcal{B}$).

4. U unitarnom prostoru $\mathcal{P}_3 = \{at^3 + bt^2 + ct + d \,|\, a,b,c,d \in \mathbb{R}\}$ polinoma stepena ≤ 3 sa skalarnim proizvodom

$$\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t$$

dat je potprostor $\mathcal{M} = \mathrm{span}\{1+t, 1\}$. Odredite jednu bazu za $\mathcal{M}^\perp$.

Važno: Ovaj papir treba predati zajedno s rješenjima zadataka! Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

(#) Dokazati da je $V = \{A \in Mat_{2\times 2}(\mathbb{R}) \mid \mathrm{trag}\, A = 0\}$ vektorski podprostor prostora $Mat_{2\times 2}(\mathbb{R})$ (gdje je $\mathrm{trag}\, A$ = sumi dijagonalnih elemenata matrice A). Odredite mu bazu i dimenziju. Nadopunite nađenu bazu do baze za $Mat_{2\times 2}(\mathbb{R})$.

Rj.

Primjetimo da je

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+d=0,\ a,b,c,d \in \mathbb{R} \right\} =$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat_{2\times 2}(\mathbb{R}) : \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \right\}$$

Pokažimo prvo da je V vektorski podprostor od $Mat_{2\times 2}(\mathbb{R})$, tj. pokažimo da je V neprazan i da vrijede osobine (A1) i (M1):

(A1) $A, B \in V \Rightarrow A+B \in V$

(M1) $A \in V \Rightarrow \alpha A \in V$ za $\forall \alpha \in \mathbb{R}$.

V je neprazan npr. $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix} \in V$ $(-1+1=0)$

Neka su A, i B dvije proizvoljne matrice iz V npr. $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ i $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Tada $a_1 + a_4 = 0$ i $b_1 + b_4 = 0$.

Kako je $A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$ to je i $a_1+b_1+a_4+b_4 = 0$ pa je $A+B \in V$, $(\mathrm{trag}(A+B)=0)$. Slično $\alpha A = \begin{bmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{bmatrix} \Rightarrow$

$\Rightarrow \mathrm{trag}(\alpha A) = \alpha a_1 + \alpha a_4 = \alpha(a_1 + a_4) = 0 \Rightarrow \alpha A \in V$

V je vektorski podprostor prostora $Mat_{2\times 2}(\mathbb{R})$.

Primjetimo da je puno lakše odrediti bazu i dimenziju prostora $V' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a+d=0 \right\}$, pa dobijeni rezultat iskoristiti za V.

$$V' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \right\} = \ker\left(\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}\right).$$

$\bar{A} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \operatorname{rang} A = \operatorname{rang} \bar{A} = 1 < 4$

$\Rightarrow$ 3 promjenjive uzimamo proizvoljno

$a+d=0 \Rightarrow a=-d$

$$V' = \left\{ \begin{pmatrix} -s \\ t \\ v \\ s \end{pmatrix} \,\middle|\, s,t,v \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Prema tome baza za V je $\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ i V je dimenzije 3.

Nadopunimo prostor V' do baze za $\mathbb{R}^4$ pa iskoristimo taj rezultat za V.

Odredimo četiri linearno nezavisna vektora iz sljedeće matrice

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \dots \sim \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 \end{pmatrix}$$

Prema tome baza za $Mat_{2 \times 2}(\mathbb{R})$ je

$$\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

(#) Odrediti za koje vrijednosti nepoznate x će vektor $(0,1,1,4)^T \in \mathbb{R}^4$ pripadati $im(A)$ ako je

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1-x & 1 & 1 & 1 \\ 0 & 1-x & 1 & 1 \\ 0 & 0 & 1-x & 1 \end{bmatrix}.$$

Rj: $im(A) = \{Ay \mid y \in \mathbb{R}^4\}$ = prostor generisan pomoću kolona matrice A

$b \in im(A)$ akko se vektor b može izraziti kao linearna kombinacija kolona matrice A

Drugim riječima tražimo one nepoznate x za koje sistem $Ay = b$ ima bar jedno rješenje.

$$\overline{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1-x & 1 & 1 & 1 & 1 \\ 0 & 1-x & 1 & 1 & 1 \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{II_v - I_v}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -x & 0 & 0 & 0 & 1 \\ 0 & 1-x & 1 & 1 & 1 \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{(*)}{\sim}$$

Primjetimo da moramo razmotriti slučaj kada je $x = 0$

$$\overset{(*)}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \underset{III_v - IV_v}{\overset{I_v - III_v}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \overset{II_v \leftrightarrow IV_v}{\sim} \cdots \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

vidimo da sistem u ovom slučaju nema rješenja

Razmotrimo slučaj kada je $x \neq 0$

$$\underset{II_v : (\;\;)}{\overset{(*)}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1/x \\ 0 & 1-x & 1 & 1 & 1 \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{II_v + I_v}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & 1-x & 1 & 1 & 1 \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{III_v - II_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & -x & 0 & 0 & 1-\frac{1}{x} \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{III_v : x}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & -1 & 0 & 0 & \frac{1}{x}-\frac{1}{x^2} \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right]$$

$$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & \frac{1}{x} \\ 0 & 0 & 1 & 1 & \frac{2}{x}-\frac{1}{x^2} \\ 0 & 0 & 1-x & 1 & 4 \end{array}\right] \overset{IV_v - III_v}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & 0 & 1 & 1 & 2/x - 1/x^2 \\ 0 & 0 & -x & 0 & 4-\frac{2}{x}+\frac{1}{x^2} \end{array}\right]$$

$$\overset{IV_v : x}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & 0 & 1 & 1 & 2/x - 1/x^2 \\ 0 & 0 & -1 & 0 & \frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3} \end{array}\right] \overset{IV_v + III_v}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1/x \\ 0 & 0 & 1 & 1 & 2/x - 1/x^2 \\ 0 & 0 & 0 & 1 & \frac{6}{x}-\frac{3}{x^2}+\frac{1}{x^3} \end{array}\right]$$

Možemo zaključiti da za sve vrijednosti x za koje je $x \neq 0$ dati vektor $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix} \in \mathbb{R}^4$ će pripadati $\mathrm{im}(A)$.

(#) Zadano je preslikavanje $T: V^3(O) \to V^3(O)$ izrazom

$$T(a\vec{i} + b\vec{j} + c\vec{k}) = (a-2b+c)\vec{i} + 3a\vec{j} - (2a-4c)\vec{k}.$$

Dokazati da je T linearni operator i odredite mu matrični prikaz u bazi $\mathcal{B} = \{\vec{i}-\vec{j}, 2\vec{i}+\vec{j}, \vec{i}+\vec{k}\}$ (drugim riječima odredite $[T]_{\mathcal{B}}$).

Rj. Zbog jednostavnijeg zapisa umjesto vektorskog prostora $V^3(O)$ posmatrajmo prostor $\mathbb{R}^3$ i preslikavanje $T': \mathbb{R}^3 \to \mathbb{R}^3$ definisano sa

$$T'\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-2b+c \\ 3a \\ -2a+4c \end{pmatrix}$$

Pokažimo prvo da je T' linearni operator, tj. da je T' linearna f-ja sa $\mathbb{R}^3$ u $\mathbb{R}^3$. Drugim riječima da vrijedi

$$T'\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right) = T'\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + T'\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad i \quad T\left(\alpha\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \alpha T\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\forall \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3; \forall \alpha \in \mathbb{R}$$

$$T'\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right) = T'\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} = \begin{pmatrix} (a_1+b_1) - 2(a_2+b_2) + (a_3+b_3) \\ 3(a_1+b_1) \\ -2(a_1+b_1) + 4(a_3+b_3) \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 - 2a_2 + a_3 \\ 3a_1 \\ -2a_1 + 4a_3 \end{pmatrix} + \begin{pmatrix} b_1 - 2b_2 + b_3 \\ 3b_1 \\ -2b_1 + 4b_3 \end{pmatrix} = T'\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + T'\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$T'\left(\alpha\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = T'\begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix} = \begin{pmatrix} \alpha a - 2\alpha b + \alpha c \\ 3\alpha a \\ -2\alpha a + 4\alpha c \end{pmatrix} = \alpha \begin{pmatrix} a-2b+c \\ 3a \\ -2a+4c \end{pmatrix} = \alpha T'\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Time smo dokazali da je T' linearni operator. Kako iste osobine vrijede i za T, možemo zaključiti da je i T linearni operator.

Odredimo još matrični prikaz (matricu koordinata) operatora T u bazi $\mathcal{B}$. Drugim riječima odredimo ($\mathcal{B}=\{u_1,u_2,u_3\}$)

$$[T]_{\mathcal{B}} = \begin{pmatrix} | & | & | \\ [T(u_1)]_{\mathcal{B}} & [T(u_2)]_{\mathcal{B}} & [T(u_3)]_{\mathcal{B}} \\ | & | & | \end{pmatrix}$$

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Zbog jednostavnosti zapisa umjesto $\mathcal{B}=\{\vec{i}-\vec{j}, 2\vec{i}+\vec{j}, \vec{i}+\vec{k}\}$ promatrajmo $\mathcal{B}' = \left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$, i odredimo $[T']_{\mathcal{B}'}$.

$$[T']_{\mathcal{B}'} = \begin{pmatrix} | & | & | \\ \left[T'\begin{pmatrix}1\\-1\\0\end{pmatrix}\right]_{\mathcal{B}'} & \left[T'\begin{pmatrix}2\\1\\0\end{pmatrix}\right]_{\mathcal{B}'} & \left[T'\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_{\mathcal{B}'} \\ | & | & | \end{pmatrix}$$

$$T'\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1-2(-1)+0\\3\cdot(-1)\\-2\cdot 1+4\cdot 0\end{pmatrix} = \begin{pmatrix}3\\-3\\-2\end{pmatrix} \Rightarrow \dots \Rightarrow \left[T'\begin{pmatrix}1\\-1\\0\end{pmatrix}\right]_{\mathcal{B}'} = \begin{pmatrix}11/3\\2/3\\-2\end{pmatrix}$$

$$T'\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}2-2\cdot 1+0\\3\cdot 2\\-2\cdot 2+4\cdot 0\end{pmatrix} = \begin{pmatrix}0\\6\\-4\end{pmatrix} \Rightarrow \dots \Rightarrow \left[T'\begin{pmatrix}2\\1\\0\end{pmatrix}\right]_{\mathcal{B}'} = \begin{pmatrix}-8/3\\10/3\\-4\end{pmatrix}$$

$$T'\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1-2\cdot 0+1\\3\cdot 1\\-2\cdot 1+4\cdot 1\end{pmatrix} = \begin{pmatrix}2\\3\\2\end{pmatrix} \Rightarrow \dots \Rightarrow \left[T'\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_{\mathcal{B}'} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$$

Prema tome

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{11}{3} & -\frac{8}{3} & -2 \\ \frac{2}{3} & \frac{10}{3} & 1 \\ -2 & -4 & 2 \end{pmatrix}.$$

$[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}'}[u]_{\mathcal{B}'}$

Naravno, $[T]_{\mathcal{B}}$ se mogao odrediti i na drugi način. KAKO?

(#) U unitarnom prostoru $\mathcal{P}_3 = \{at^3+bt^2+ct+d \mid a,b,c,d \in \mathbb{R}\}$ polinoma stepena ≤ 3 sa skalarnim proizvodom

$$<p,q> = \int_{-1}^{1} p(t)q(t)\,dt$$

dat je potprostor $\mathcal{M} = \text{span}\{1+t, 1\}$. Odrediti jednu bazu za $\mathcal{M}^\perp$.

Rj. Prema definiciji

$$\mathcal{M}^\perp = \{q \in \mathcal{P}_3 \mid <p,q> = 0,\ p \in \mathcal{M}\}$$

Neka je $q(t) = at^3+bt^2+ct+d$, pa odredimo koeficijente a, b, c i d t.d. $<p_1,q>=0$ i $<p_2,q>=0$ gdje su $p_1(t) = 1+t$, $p_2(t)=1$.

$$<1+t,\ at^3+bt^2+ct+d> = \int_{-1}^{1}(1+t)(at^3+bt^2+ct+d)\,dt =$$

$$= \int_{-1}^{1}(at^3+bt^2+ct+d+at^4+bt^3+ct^2+dt)\,dt =$$

$$= \int_{-1}^{1}(at^4+(a+b)t^3+(b+c)t^2+(c+d)t+d)\,dt = \dots = \frac{2}{5}a+\frac{2}{3}b+\frac{2}{3}c+2d$$

$$<1,\ at^3+bt^2+ct+d> = \int_{-1}^{1}(at^3+bt^2+ct+d)\,dt = \dots = \frac{2}{3}b+2d$$

Time smo dobili sistem
$$\frac{2}{5}a+\frac{2}{3}b+\frac{2}{3}c+2d=0$$
$$\frac{2}{3}b+2d=0$$

$$\bar{A} = \left[\begin{array}{cccc|c} \frac{2}{5} & \frac{2}{3} & \frac{2}{3} & 2 & 0 \\ 0 & \frac{2}{3} & 0 & 2 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \end{array}\right]$$

$\Rightarrow$ $\operatorname{rang} \bar{A} = \operatorname{rang} A = 2 < 4 \Rightarrow$ dvije promjenjive uzimamo proizvoljno

npr. $x_3 = s \quad x_4 = t$ tj.

$c = s, \quad d = t$

$\Rightarrow \quad a = -\frac{5}{3}s, \quad b = -3t$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -\frac{5}{3}s \\ -3t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} t$$

$$M^{\perp} = \operatorname{span}\left\{ -\frac{5}{3}t^3 + t, \ -3t^2 + 1 \right\}$$

(baza za $M^{\perp}$ je $\left\{ \left(-\frac{5}{3}\right)t^3 + t, \ -3t^2 + 1 \right\}$)